Exercise Sheet 5

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Let us consider $\mathcal{A}$, $\mathcal{B}$ two abelian categories.

1. Let $\mathcal{S}$ be the category of short exact sequences in $\mathcal{A}$, and δ a homological δ-functor with additive functors $\{T_i : \mathcal{A} \to \mathcal{B}\}_{i\geq 0}$.
   For $i \geq 0$ we call F_i the functor $\mathcal{S} \to \mathcal{B}$ which sends the short exact sequence

   $$0 \to A \to B \to C \to 0$$

   onto the object $T_i(C)$, and the morphism $f_\bullet$

   $$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f_2"] & B \arrow[r] \arrow[d, "f_1"] & C \arrow[r] \arrow[d, "f_0"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

   onto the morphism $T_i(f_0)$ in $\mathcal{B}$.
   Similarly we define the functor $G_i : \mathcal{S} \to \mathcal{B}$ which sends the short exact sequence

   $$0 \to A \to B \to C \to 0$$

   onto the object $T_i(A)$, and the morphism $f_\bullet$

   $$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f_2"] & B \arrow[r] \arrow[d, "f_1"] & C \arrow[r] \arrow[d, "f_0"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

   onto the morphism $T_i(f_2)$ in $\mathcal{B}$.
   These are both well defined functors since each T_i is a functor.
   Since for each $i \geq 0$, δ_i assigns to each object

   $$0 \to A \to B \to C \to 0$$

   in $\mathcal{S}$ a morphism $T_i(C) \to T_{i-1}(A)$, for δ_i to be a natural transformation $F_i \Rightarrow G_{i-1}$ it remains to show naturality, which is to say that for all morphisms of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f_2"] & B \arrow[r] \arrow[d, "f_1"] & C \arrow[r] \arrow[d, "f_0"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

the diagram

$$\begin{tikzcd} T_i(C) \arrow[r, "\delta_i(0\to A\to B\to C\to 0)"] \arrow[d, "T_i(f_2)=F_i(f_\bullet)"'] & T_{i-1}(A) \arrow[d, "T_{i-1}(f_0)=G_{i-1}(f_\bullet)"] \\ T_i(C') \arrow[r, "\delta_i(0\to A'\to B'\to C'\to 0)"'] & T_{i-1}(A') \end{tikzcd}$$

commutes. This is the case because δ is a homological δ-functor which allows us to conclude.

2. Suppose first that $P_\bullet$ is a projective object in $Ch(\mathcal{A})$.
We then have that $P_\bullet[-1]$ is also projective because for any diagram

$$\begin{tikzcd} & P_\bullet[-1] \arrow[d] \\ A_\bullet \arrow[r, two heads] & B_\bullet \end{tikzcd}$$

the projectiveness of $P_\bullet$ means we have a morphism which makes the induced diagram

$$\begin{tikzcd} & P_\bullet \arrow[d] \arrow[dl, dashed] \\ A_\bullet[+1] \arrow[r, two heads] & B_\bullet[+1] \end{tikzcd}$$

commute. Taking the induced morphism $P_\bullet[-1] \to A_\bullet$ makes the diagram

$$\begin{tikzcd} & P_\bullet[-1] \arrow[d] \arrow[dl, dashed] \\ A_\bullet \arrow[r, two heads] & B_\bullet \end{tikzcd}$$

commute.

We can then consider the diagram

$$\begin{tikzcd} & P_\bullet[-1] \arrow[d, "Id_{P_\bullet[-1]}"] \\ Cone(P_\bullet) \arrow[r, two heads, "\pi^1_\bullet"] & P_\bullet[-1] \end{tikzcd}$$

where the morphism $\pi^1_\bullet$ is projection onto the first element

$$\pi^1_n : Cone(P_\bullet)_n = P_{n-1} \oplus P_n \to P_{n-1} = P_\bullet[-1]_n,$$

and similarly $\pi^2_\bullet$ as used below is projection onto the second element.
The projectiveness of $P_\bullet[-1]$ means we have a morphism of chain complexes

$$\psi_\bullet : P_\bullet[-1] \to Cone(P_\bullet)$$

such that $\pi^1_\bullet \circ \psi_\bullet = Id_{P_\bullet[-1]}$.

This identity and $\psi_\bullet$ being a chain morphism yield the following equalities for all $n \in \mathbb{Z}$:

$$\begin{aligned} \pi^1_n \circ \psi_n &= Id_{P_{n-1}} \\ \psi_n \circ d^P_n &= d^{Cone(P)}_{n+1} \circ \psi_{n+1}. \end{aligned}$$

Using the definition of the chain maps for the cone of $P_\bullet$ and plugging in the first equation gives

$$\begin{aligned} \pi^2_n \circ \psi_n \circ d^P_n &= \pi^2_n \circ d^{Cone(P)}_{n+1} \circ \psi_{n+1} \\ &= \pi^2_n \circ \begin{bmatrix} -d^P_n & 0 \\ Id_{P_n} & d^P_{n+1} \end{bmatrix} \circ \psi_{n+1} \\ &= \begin{bmatrix} Id_{P_n} & d^P_{n+1} \end{bmatrix} \circ \psi_{n+1} \\ &= \pi^1_{n+1} \circ \psi_{n+1} + d^P_{n+1} \circ \pi^2_{n+1} \circ \psi_{n+1} \\ &= Id_{P_n} + d^P_{n+1} \circ \pi^2_{n+1} \circ \psi_{n+1} \end{aligned}$$

so

$$Id_{P_n} = (\pi^2_n \circ \psi_n) \circ d^P_n - d^P_{n+1} \circ (\pi^2_{n+1} \circ \psi_{n+1})$$

which means that the identity is nullhomotopic, and $P_\bullet$ is split exact.

To show that each P_n is projective we fix $n \in \mathbb{Z}$ and consider any diagram

$$\begin{tikzcd} & P_n \arrow[d, "f"] \\ A \arrow[r, "\pi"] & B \end{tikzcd}$$

where π is epi, and then construct the morphism of chain complexes

$$\begin{tikzcd} \cdots \arrow[r, dotted] & 0 \arrow[r] \arrow[d] & A \arrow[r, "Id_A"] \arrow[d, "\pi"] & A \arrow[r] \arrow[d, "\pi"] & 0 \arrow[r, dotted] \arrow[d] & \cdots \\ \cdots \arrow[r, dotted] & 0 \arrow[r] & B \arrow[r, "Id_B"'] & B \arrow[r] & 0 \arrow[r, dotted] & \cdots \end{tikzcd}$$

where the chain complexes on top and bottom have non-zero elements only in the $(n+1)$-th and n-th positions, noting that it is epi since π is. We also have a morphism of chain complexes

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & P_{n+2} \arrow[r, "d^P_{n+2}"] \arrow[d] & P_{n+1} \arrow[r, "d^P_{n+1}"] \arrow[d, "f\circ d^P_{n+1}"] & P_n \arrow[r, "d^P_n"] \arrow[d, "f"] & P_{n-2} \arrow[r, dotted] \arrow[d] & \cdots \\
\cdots \arrow[r, dotted] & 0 \arrow[r] & B \arrow[r, "Id_B"'] & B \arrow[r] & 0 \arrow[r, dotted] & \cdots
\end{tikzcd}$$

which allows us to use the projectiveness of $P_\bullet$ to obtain a morphism of chain complexes $g_\bullet$ such that the diagram

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & P_{n+2} \arrow[r, "d^P_{n+2}"] \arrow[d] & P_{n+1} \arrow[r, "d^P_{n+1}"] \arrow[d, "g_{n+1}"] \arrow[dd, bend right, "f\circ d^P_{n+1}"'] & P_n \arrow[r, "d^P_n"] \arrow[d, "g_n"] \arrow[dd, bend right, "f"'] & P_{n-2} \arrow[r, dotted] \arrow[d] & \cdots \\
\cdots \arrow[r, dotted] & 0 \arrow[r] \arrow[d] & A \arrow[r, "Id_A"'] \arrow[d, "\pi"] & A \arrow[r] \arrow[d, "\pi"] & 0 \arrow[r, dotted] \arrow[d] & \cdots \\
\cdots \arrow[r, dotted] & 0 \arrow[r] & B \arrow[r, "Id_B"] & B \arrow[r] & 0 \arrow[r, dotted] & \cdots
\end{tikzcd}$$

commutes. This directly implies the commutativity of the diagram

$$\begin{tikzcd}
 & P_n \arrow[dl, "g_n"'] \arrow[d, "f"] \\
A \arrow[r, "\pi"'] & B
\end{tikzcd}$$

and the arbitrary choice of $n \in \mathbb{Z}$ allows us to conclude.

We have now shown that

$$\begin{aligned}
&P_\bullet \text{ is a projective object in } Ch(\mathcal{A}) \\
&\qquad\qquad \implies P_\bullet \text{ is split exact and each } P_n \text{ is projective in } \mathcal{A}
\end{aligned}$$

and go about about proving the converse.

Let $P_\bullet$ be a chain complex in $Ch(\mathcal{A})$ which is split exact and such that each P_n is projective in $\mathcal{A}$.

Option 1:
$P_\bullet$ being split exact means we have morphisms $\psi_n : P_n \to P_{n+1}$ such that

$$\psi_{n-1} \circ d^P_n + d^P_{n+1} \circ \psi_n = Id_{P_n}$$

for all $n \in \mathbb{Z}$.
Let us consider two chain morphisms $f_\bullet$ and $\pi_\bullet$

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & A_1 \arrow[r, "d_1^A"] \arrow[d, "\pi_1"] & A_0 \arrow[r, "d_0^A"] \arrow[d, "\pi_0"] & A_{-1} \arrow[r, dotted] \arrow[d, "\pi_{-1}"] & \cdots \\
\cdots \arrow[r, dotted] & B_1 \arrow[r, "d_1^B"'] & B_0 \arrow[r, "d_0^B"'] & B_{-1} \arrow[r, dotted] & \cdots \\
\cdots \arrow[r, dotted] & P_1 \arrow[r, "d_1^P"'] \arrow[u, "f_1"] & P_0 \arrow[r, "d_0^P"'] \arrow[u, "f_0"] & P_{-1} \arrow[r, dotted] \arrow[u, "f_{-1}"] & \cdots
\end{tikzcd}$$

where $\pi_\bullet$ is epi.
The projectiveness of each P_n means we have morphisms $h_n : P_n \to A_n$ which make the diagram

$$\begin{tikzcd}
 & P_n \arrow[dl, "h_n"'] \arrow[d, "f_n"] \\
A_n \arrow[r, "\pi_n"'] & B_n
\end{tikzcd}$$

commute for each $n \in \mathbb{Z}$.
We claim that setting

$$g_n \coloneqq d_{n+1}^A \circ h_{n+1} \circ \psi_n + h_n \circ \psi_{n-1} \circ d_n^P : P_n \to A_n$$

for each $n \in \mathbb{Z}$ will give the identity

$$f_\bullet = \pi_\bullet \circ g_\bullet.$$

This is easily verified since for all $n \in \mathbb{Z}$ we have

$$\begin{aligned}
\pi_n \circ g_n &= \pi_n \circ (d_{n+1}^A \circ h_{n+1} \circ \psi_n + h_n \circ \psi_{n-1} \circ d_n^P) \\
&= d_{n+1}^B \circ \pi_{n+1} \circ h_{n+1} \circ \psi_n + f_n \circ \psi_{n-1} \circ d_n^P \\
&= d_{n+1}^B \circ f_{n+1} \circ \psi_n + f_n \circ \psi_{n-1} \circ d_n^P \\
&= f_n \circ (\psi_{n-1} \circ d_n^P + d_{n+1}^P \circ \psi_n) \\
&= f_n \circ Id_{P_n} \\
&= f_n
\end{aligned}$$

by using the fact that ψ_n, ψ_{n-1} are homotopy maps.

Additionally, $g_\bullet$ is a well-defined chain morphism because

$$\begin{aligned}
d_n^A \circ g_n &= d_n^A \circ (d_{n+1}^A \circ h_{n+1} \circ \psi_n + h_n \circ \psi_{n-1} \circ d_n^P) \\
&= d_n^A \circ h_n \circ \psi_{n-1} \circ d_n^P \\
&= (d_n^A \circ h_n \circ \psi_{n-1} + h_{n-1} \circ \psi_{n-2} \circ d_{n-1}^P) \circ d_n^P \\
&= g_{n-1} \circ d_n^P
\end{aligned}$$

for all $n \in \mathbb{Z}$.

Since the chain complex $A_\bullet$ was chosen arbitrarily we conclude that $P_\bullet$ is projective.

Option 2:
$P_\bullet$ being split exact means we have morphisms $\psi_n : P_n \to P_{n+1}$ such that

$$\psi_{n-1} \circ d_n^P + d_{n+1}^P \circ \psi_n = Id_{P_n}$$

for all $n \in \mathbb{Z}$.
The existence of such morphisms means that the chain complex is exact and also allows us to construct the isomorphism

$$\begin{bmatrix} d_{n+1}^P \circ \psi_n \\ d_n^P \end{bmatrix} : P_n \to \ker(d_n^P) \oplus \operatorname{im}(d_n^P)$$

with inverse

$$\begin{bmatrix} \iota_n & \psi_{n-1} \end{bmatrix} : \ker(d_n^P) \oplus \operatorname{im}(d_n^P) \to P_n$$

for all $n \in \mathbb{Z}$.
We would like to show that

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & P_{n+1} \arrow[r, "d_{n+1}^P"] \arrow[d, "\begin{bmatrix} d_{n+2}^P \circ \psi_{n+1} \\ d_{n+1}^P \end{bmatrix}"] & P_n \arrow[r, "d_n^P"] \arrow[d, "\begin{bmatrix} d_{n+1}^P \circ \psi_n \\ d_n^P \end{bmatrix}"] & P_{n-1} \arrow[r, dotted] \arrow[d, "\begin{bmatrix} d_n^P \circ \psi_{n-1} \\ d_{n-1}^P \end{bmatrix}"] & \cdots \\
\cdots \arrow[r, dotted] & \ker(d_{n+1}^P) \oplus \operatorname{im}(d_{n+1}^P) \arrow[r] & \ker(d_n^P) \oplus \operatorname{im}(d_n^P) \arrow[r] & \ker(d_{n-1}^P) \oplus \operatorname{im}(d_{n-1}^P) \arrow[r, dotted] & \cdots
\end{tikzcd}$$

is an isomorphism of chain complexes, where the differentials of the chain on the bottom are all of the form $\begin{bmatrix} 0 & \phi_n \\ 0 & 0 \end{bmatrix}$, with ϕ_n being isomorphisms given by the exactness of the chain complex $P_\bullet$.
This is the case because by choosing the ϕ_n appropriately, we can use Freyd-Mitchell's

embedding theorem and suppose that ϕ_n is the identity, which then means

$$\begin{aligned}
\begin{bmatrix} d_n^P \circ \psi_{n-1} \\ d_{n-1}^P \end{bmatrix} \circ d_n^P &= \begin{bmatrix} d_n^P \circ \psi_{n-1} \circ d_n^P \\ d_{n-1}^P \circ d_n^P \end{bmatrix} \\
&= \begin{bmatrix} d_n^P \circ (\mathrm{Id}_{P_n} - d_{n+1}^P \circ \psi_n) \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} d_n^P \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \phi_n \circ d_n^P \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \phi_n \\ 0 & 0 \end{bmatrix} \circ \begin{bmatrix} d_{n+1}^P \circ \psi_n \\ d_n^P \end{bmatrix}.
\end{aligned}$$

Finally, there is an isomorphism between the chain complex on the bottom row and the direct sum over every $n \in \mathbb{Z}$ of the chain complexes $P(n)$

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & 0 \arrow[r] & \mathrm{im}(d_n^P) \arrow[r, "\phi_n"] & \ker(d_{n-1}^P) \arrow[r] & 0 \arrow[r, dotted] & \cdots
\end{tikzcd}$$

n-th position in the chain complex

because the former satisfies the universal properties of the product and coproduct.

This means that it is enough to show that the direct sum is projective, for which it is enough to show that each individual element in the direct sum is projective.

To this end, fix $n \in \mathbb{Z}$ and suppose we have morphisms of chain complexes

$$\begin{tikzcd}
 & P(n) \arrow[d, "f_\bullet"] \\
A \arrow[r, "\pi_\bullet"'] & B
\end{tikzcd}$$

where $\pi_\bullet$ is epi.

Since $P_n \cong \ker(d_n^P) \oplus \mathrm{im}(d_n^P)$ is projective, $\mathrm{im}(d_n^P)$ must be as well, so there exists a morphism $g_n : \mathrm{im}(d_n^P) \to A_n$ such that $\pi_n \circ g_n = f_n$. Now setting $g_{n-1} = d_n^A \circ g_n \circ \phi_n^{-1}$, and completing $g_\bullet : P(n) \to A_\bullet$ everywhere else with zero morphisms we have that $g_\bullet$ is a well defined morphism of chain complexes by definition of g_{n-1}, and satisfies

$\pi_\bullet \circ g_\bullet = f_\bullet$. At every index other that $n-1$ this is immediate, and at $n-1$ we have

$$\begin{aligned}\pi_{n-1} \circ g_{n-1} &= \pi_{n-1} \circ d_n^A \circ g_n \circ \phi_n^{-1} \\ &= d_n^B \circ \pi_n \circ g_n \circ \phi_n^{-1} \\ &= d_n^B \circ f_n \circ \phi_n^{-1} \\ &= f_{n-1}\end{aligned}$$

because $\pi_\bullet$ and $f_\bullet$ are both morphisms of chain complexes.

We therefore have that each $P(n)$ is projective which allows us to conclude.

3. Suppose that $\mathcal{A}$ has enough projectives, and let $A_\bullet$ be a chain complex in $Ch(\mathcal{A})$. For all $n \in \mathbb{Z}$ we have a projective object P_n in $\mathcal{A}$ and an epi $f_n : P_n \to A_n$.
These projective objects define a chain complex

$$\begin{tikzcd} {} \arrow[r, dotted] & P_2 \oplus P_1 \arrow[r] & P_1 \oplus P_0 \arrow[r] & P_0 \oplus P_{-1} \arrow[r, dotted] & {} \end{tikzcd}$$

with differentials $\begin{bmatrix} 0 & Id \\ 0 & 0 \end{bmatrix}$.
Setting $Q_n := P_{n+1} \oplus P_n$ we may call this chain complex $Q_\bullet$, and we claim that setting

$$g_n := \begin{bmatrix} d_{n+1}^A \circ f_{n+1} & f_n \end{bmatrix} : Q_n \to A_n$$

for all $n \in \mathbb{Z}$ defines a chain morphism $g_\bullet : Q_\bullet \to A_\bullet$.
This is the case because

$$\begin{aligned} d_n^A \circ g_n &= d_n^A \circ \begin{bmatrix} d_{n+1}^A \circ f_{n+1} & f_n \end{bmatrix} \\ &= \begin{bmatrix} d_n^A \circ d_{n+1}^A \circ f_{n+1} & d_n^A \circ f_n \end{bmatrix} \\ &= \begin{bmatrix} 0 & d_n^A \circ f_n \end{bmatrix} \\ &= \begin{bmatrix} 0 & d_n^A \circ f_n \circ Id_{P_n} \end{bmatrix} \\ &= \begin{bmatrix} d_n^A \circ f_n & f_{n-1} \end{bmatrix} \circ \begin{bmatrix} 0 & Id_{P_n} \\ 0 & 0 \end{bmatrix} \\ &= g_{n-1} \circ d_n^Q. \end{aligned}$$

Moreover, each map g_n is epi, because its second component, f_n, is epi. Using Freyd-Mitchell it is easy to see that any map from a direct sum of two objects to a third object is epi if either of its two components are epi. Therefore $g_\bullet$ is epi in $Ch(\mathcal{A})$.
The maps $\psi_n : Q_n = P_{n+1} \oplus P_n \to P_{n+2} \oplus P_{n+1} = Q_{n+1}$ given by $\begin{bmatrix} 0 & 0 \\ Id & 0 \end{bmatrix}$ are homotopy maps between the identity and 0 on $Q_\bullet$, because

$$d_{n+1}^Q \circ \psi_n = \begin{bmatrix} Id & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\psi_{n-1} \circ d_n^Q = \begin{bmatrix} 0 & 0 \\ 0 & Id \end{bmatrix}$$

so,

$$Id_{Q_n} = \psi_{n-1} \circ d_n^Q + d_{n+1}^Q \circ \psi_n.$$

Finally, using the result of the previous exercise, it remains to show that each $Q_n = P_{n+1} \oplus P_n$ is projective for us to have defined a projective chain complex $Q_\bullet$ and an epi $g_\bullet : Q_\bullet \to A_\bullet$.
This is immediate since each P_n is projective.

4. Suppose we have the following diagram in $\mathcal{A}$

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
\cdots \arrow[r, dotted] & P_2' \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r, "\epsilon'"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \\
 & & & & A \arrow[d, "\pi_A"] & \\
\cdots \arrow[r, dotted] & P_2'' \arrow[r] & P_1'' \arrow[r] & P_0'' \arrow[r, "\epsilon''"] & A'' \arrow[r] \arrow[d] & 0 \\
 & & & & 0 &
\end{tikzcd}$$

where the column is exact and the rows are projective resolutions.
We set $P_i := P_i' \oplus P_i''$ and use the canonical injection and projection associated to the direct sum to extend the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r, dotted] & P_2' \arrow[r] \arrow[d, "\iota_2"] & P_1' \arrow[r] \arrow[d, "\iota_1"] & P_0' \arrow[r, "\epsilon'"] \arrow[d, "\iota_0"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \\
 & P_2 \arrow[d, "\pi_2"] & P_1 \arrow[d, "\pi_1"] & P_0 \arrow[d, "\pi_0"] & A \arrow[d, "\pi_A"] & \\
\cdots \arrow[r, dotted] & P_2'' \arrow[r] \arrow[d] & P_1'' \arrow[r] \arrow[d] & P_0'' \arrow[r, "\epsilon''"] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

noting that the columns are all exact because $\mathcal{A}$ is abelian.
We will now construct the chain maps for $P_\bullet$ by induction such that $\iota_\bullet$ becomes a

chain map $P'_\bullet \to P_\bullet$ and $\pi_\bullet$ becomes a chain map $P_\bullet \to P''_\bullet$, and $P_\bullet$ is a projective resolution of A.
In fact we only have to guarantee the first two conditions because by the long exact sequence of homology groups of

$$0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$$

any such chain maps would make the sequence exact.

At step zero we define $\epsilon : P_0 \to A$ by setting $\epsilon = \begin{bmatrix} f'_0 & f''_0 \end{bmatrix}$, where $f'_0 = \iota_A \circ \epsilon'$, and f''_0 is given by the projectiveness of P''_0, making the diagram

$$\begin{tikzcd} & A \arrow[d, "\pi_A"] \\ P''_0 \arrow[ur, dashed, "f''_0"] \arrow[r, "\epsilon''"'] & A'' \end{tikzcd}$$

commute (π_A is epi because the column is exact).
For $n \geq 1$, we write $d_n^P = \begin{bmatrix} f'_n & f''_n \\ g'_n & g''_n \end{bmatrix}$.
Considering the diagram

$$\begin{tikzcd} P'_n \arrow[r, "d_n^{P'}"] \arrow[d, "\iota_n"'] & P'_{n-1} \arrow[d, "\iota_{n-1}"] \\ P_n \arrow[r, "d_n^P"] \arrow[d, "\pi_n"'] & P_{n-1} \arrow[d, "\pi_{n-1}"] \\ P''_n \arrow[r, "d_n^{P''}"] & P''_{n-1} \end{tikzcd}$$

we observe that the top square commutes if and only if $f'_n = d_n^{P'}$ and $g'_n = 0$. If this is the case then the bottom square commutes if and only if $g''_n = d_n^{P''}$. This means that the diagram commutes if and only if $d_n^P = \begin{bmatrix} d_n^{P'} & f''_n \\ 0 & d_n^{P''} \end{bmatrix}$ for some morphism f''_n.

We define the morphisms f''_n by induction such that $d_n^P \circ d_{n-1}^P = 0$.

If $n = 1$ then we have $d_n^P \circ d_{n-1}^P = \begin{bmatrix} f'_0 \circ d_1^{P'} & f'_0 \circ f''_1 + f_0 \circ d_1^{P''} \end{bmatrix}$. Given that $f'_0 = \iota_A \circ \epsilon'$ and that ϵ' and $d_1^{P'}$ and consecutive maps in a projective resolution, we need f''_n such that

$$f'_0 \circ f''_n = -f''_0 \circ d_1^{P''}.$$

We notice that by definition of f_0'' we have

$$\begin{aligned}\pi_A \circ (-f_0'' \circ d_1^{P''}) &= -\pi_A \circ f_0'' \circ d_1^{P''} \\ &= -\epsilon'' \circ d_1^{P''} \\ &= 0\end{aligned}$$

so by exactness of the column in our very first diagram we have a diagram

$$\begin{tikzcd} P_1'' \arrow[d, dashed, "f_n''"'] \arrow[dr, "-f_0''\circ d_1^{P''}"] & \\ P_0' \arrow[r, "f_0'=\iota_A\circ\epsilon'"'] & \mathrm{im}(\iota) \end{tikzcd}$$

that is completed by a morphism f_n'' by projectiveness of P_1'', since the bottom morphism is epi because ϵ' is.

Suppose now that the morphisms f_{n-1}'', ..., f_1'' have all been constructed, for $n > 1$.

We have $d_n^P \circ d_{n-1}^P = \begin{bmatrix} d_{n-1}^{P'} \circ d_n^{P'} & d_{n-1}^{P'} \circ f_n'' + f_{n-1}'' \circ d_n^{P''} \\ 0 & d_{n-1}^{P''} \circ d_n^{P''} \end{bmatrix} = \begin{bmatrix} 0 & d_{n-1}^{P'} \circ f_n'' + f_{n-1}'' \circ d_n^{P''} \\ 0 & 0 \end{bmatrix}$, so we want to find f_n'' such that

$$d_{n-1}^{P'} \circ f_n'' = -f_{n-1}'' \circ d_n^{P''}.$$

By the inductive argument we have that

$$\begin{aligned}d_{n-2}^{P'} \circ (-f_{n-1}'' \circ d_n^{P''}) &= -(d_{n-2}^{P'} \circ f_{n-1}'') \circ d_n^{P''} \\ &= -(-f_{n-2}'' \circ d_{n-1}^{P''}) \circ d_n^{P''} \\ &= f_{n-2}'' \circ 0 \\ &= 0\end{aligned}$$

if $n > 2$, and if $n = 2$ we have

$$\begin{aligned}\iota_A \circ (d_0^{P'} \circ (-f_1'' \circ d_2^{P''})) &= -(\iota_A \circ d_0^{P'} \circ f_1'') \circ d_2^{P''} \\ &= -(-f_0'' \circ d_1^{P''}) \circ d_2^{P''} \\ &= f_0'' \circ 0 \\ &= 0\end{aligned}$$

which means $d_{n-2}^{P'} \circ (-f_{n-1}'' \circ d_n^{P''}) = 0$ since ι_A is mono.

Therefore by projectiveness of P_n'' we can complete the diagram

$$\begin{tikzcd} P_n'' \arrow[d, dashed, "f_n''"'] \arrow[dr, "-f_{n-1}''\circ d_n^{P''}"] & \\ P_{n-1}' \arrow[r, "d_{n-1}^{P'}"'] & \ker(d_{n-2}^{P'}) \end{tikzcd}$$

with a morphism f''_n such that it commutes.

This way we can define all the chain maps, which allows us to conclude our proof of the Horseshoe Lemma.